Math 129
Fall 2018 - Spring 2019

Final Exam Review Packet

Section 1.1 Problems

The following questions can be used as a review for Math 129. These questions are not actual samples of questions that will appear on the final exam, but they will provide additional practice for the material that will be covered on the final exam. When solving these problems keep the following in mind: Full credit for correct answers will only be awarded if all work is shown. Exact simplified values must be given unless an approximation is required. Credit will not be given for an approximation when an exact value can be found by techniques covered in the course.

1. Suppose the rate at which people get a particular disease (measured in people per month) can be modeled by
$$r(t) = 10\pi \sin\left(\frac{\pi}{3}t\right) + 30.$$
Find the total number of people who will get the disease during the first three months ($0 \le t \le 3$).

   Solution: $\displaystyle\int_0^3 \left(10\pi \sin\left(\frac{\pi}{3}t\right) + 30\right) dt = 150$ people

2. If $\displaystyle\int_1^3 f(w)\, dw = 7$, find the value of $\displaystyle\int_1^2 f(5-2x)\, dx$.

   Solution: Let $u = 5 - 2x$ and change the endpoints:
$$\int_1^2 f(5-2x)\, dx = \frac{7}{2}.$$

3. Evaluate

(a) $\int \frac{t}{\sqrt{t+1}}dt$

Solution: By the method of substitution with $u = t+1$:

$$\int \frac{t}{\sqrt{t+1}}dt = \frac{2}{3}(t+1)^{3/2} - 2(t+1)^{1/2} + c.$$

You can also use integration by parts with $u = t$ and $v' = (t+1)^{-1/2}$ and we get an equivalent result (written in a different form):

$$\int \frac{t}{\sqrt{t+1}}dt = 2t(t+1)^{1/2} - \frac{4}{3}(t+1)^{3/2} + c.$$

(b) $\int \left(\frac{1}{z^2} + A\right)^2 dz$

Solution: Distribute:

$$\int \left(\frac{1}{z^2} + A\right)^2 dz = -\frac{1}{3z^3} - \frac{2A}{z} + A^2 z + c.$$

(c) $\int 3^x e^x dx$

Solution: Rewrite $3^x e^x = (3e)^x$:

$$\int 3^x e^x dx = \frac{1}{\ln(3e)}(3e)^x + c.$$

(d) $\int_0^1 \frac{\arctan y}{1+y^2}dy$

Solution: Substitution and change endpoints. Always simplify answer:

$$\int_0^1 \frac{\arctan y}{1+y^2}dy = \frac{\pi^2}{32}.$$

4. Evaluate

(a) $\displaystyle\int \frac{\ln(z^2+1)}{z^2}dz$

Solution: Let $u = \ln(z^2+1)$ and $v' = \dfrac{1}{z^2}$:

$$\int \frac{\ln(z^2+1)}{z^2}dz = -\frac{1}{z}\ln(z^2+1) + 2\arctan(z) + c.$$

(b) $\displaystyle\int x\arcsin(x^2)\,dx$

Solution: First make a substitution with $w = x^2$, then let $u = \arcsin(w)$ and $v' = 1$:

$$\int x\arcsin(x^2)\,dx = \frac{1}{2}x^2\arcsin(x^2) + \frac{1}{2}\sqrt{1-x^4} + c.$$

(c) $\displaystyle\int_0^1 x\cdot g''(x)\,dx$ where g is twice differentiable with $g(0) = 6$, $g(1) = 5$, and $g'(1) = 2$.

Solution: Let $u = x$ and $v' = g''$ for the first integration by parts:

$$\int_0^1 x\cdot g''(x)\,dx = 3.$$

5. Evaluate (you will receive a copy of the integral table during the final exam):

(a) $\int \cos^2(3\theta+2)\,d\theta$

Solution: Let $u = 3\theta + 2$ before using table formula #18.

$$\int \cos^2(3\theta+2)\,d\theta = \frac{1}{6}\cos(3\theta+2)\sin(3\theta+2) + \frac{1}{6}(3\theta+2) + c.$$

If you use another approach, your answer will look different.

(b) $\int \frac{2}{4t^2-9}dt$

Solution: Let $u = 2t$ and factor the denominator before using table formula #26.

$$\int \frac{2}{4t^2-9}dt = \frac{1}{6}\left(\ln|2t-3| - \ln|2t+3|\right) + c.$$

If you use another approach, your answer will look different.

(c) $\int \frac{dy}{\sqrt{y^2+8y+15}}$

Solution: Complete the square before using table formula #29.

$$\int \frac{dy}{\sqrt{y^2+8y+15}} = \ln\left|(y+4) + \sqrt{y^2+8y+15}\right| + c.$$

(d) $\int \frac{\sin(4\alpha)\,d\alpha}{\cos^2(4\alpha) - \cos(4\alpha)}$

Solution: Let $u = \cos(4\alpha)$ and factor the denominator before using table formula #26.

$$\int \frac{\sin(4\alpha)\,d\alpha}{\cos^2(4\alpha) - \cos(4\alpha)} = \frac{1}{4}\left(\ln|\cos(4\alpha)| - \ln|\cos(4\alpha) - 1|\right) + c.$$

6. Evaluate

(a) $\displaystyle\int \frac{3y^3+5y-1}{y^3+y}dy$

Solution: First do long division, then use partial fractions $\dfrac{A}{y}+\dfrac{By+C}{y^2+1}$:

$$\int \frac{3y^3+5y-1}{y^3+y}dy = 3y - \ln|y| + \frac{1}{2}\ln|y^2+1| + 2\arctan(y) + c.$$

(b) $\displaystyle\int \frac{5z-28}{6z^2+z-40}dz$

Solution: Use partial fractions $\dfrac{A}{3z+8}+\dfrac{B}{2z-5}$:

$$\int \frac{5z-28}{6z^2+z-40}dz = \frac{4}{3}\ln|3z+8| - \frac{1}{2}\ln|2z-5| + c.$$

(c) $\displaystyle\int \frac{dx}{(5-x^2)^{3/2}}$

Solution: Let $x=\sqrt{5}\sin(\theta)$:

$$\int \frac{dx}{(5-x^2)^{3/2}} = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + c.$$

(d) $\displaystyle\int \frac{dt}{t^2\sqrt{1+t^2}}$

Solution: Let $t=\tan(\theta)$:

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = -\frac{\sqrt{1+t^2}}{t} + c.$$

7. Let f be a differentiable function with the following values:

x	0	1	e	3
$f(x)$	5	7	10	11
$f'(x)$	2	4	9	12

Evaluate the integrals.

(a) $\int_1^3 f'(x)e^{f(x)}dx$

Solution: Let $u = f(x)$:

$$\int_1^3 f'(x)e^{f(x)}dx = e^{11} - e^7.$$

(b) $\int_1^e \frac{f'(\ln x)}{x}dx$

Solution: Let $u = \ln x$:

$$\int_1^e \frac{f'(\ln x)}{x}dx = 2.$$

8. In the study of probability, a quantity called the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Find $E(X)$ if

$$f(x) = \begin{cases} \frac{1}{7}e^{-x/7} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Solution: Use integration by parts or the table of integrals. Remember to use proper notation.

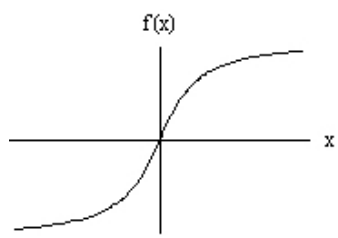
$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{0} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx = 0 + \int_{0}^{\infty} x \left(\frac{1}{7}e^{-x/7}\right) dx = 7.$$

9. Find an approximation of $\int_0^1 e^{-t^2} dt$ using the midpoint rule with $n = 2$. (Show your work).

Solution:

$$\int_0^1 e^{-t^2} dt \approx \frac{1}{2}e^{-1/16} + \frac{1}{2}e^{-9/16} \approx 0.75459794$$

10. Below is the graph of a function $y = f(x)$. Assume the function is increasing for $-\infty < x < \infty$, concave up for $-\infty < x < 0$, and concave down for $0 < x < \infty$. Which of the following are true for any number of subdivisions? Select all that apply.



(a) Left$(n) < \int_{-15}^{11} f(x)\, dx <$ Right(n).

(b) Right$(n) < \int_{-15}^{11} f(x)\, dx <$ Left(n).

(c) Trap$(n) < \int_{-15}^{0} f(x)\, dx <$ Mid(n).

(d) Mid$(n) < \int_{-15}^{0} f(x)\, dx <$ Trap(n).

(e) Trap$(n) < \int_{0}^{11} f(x)\, dx <$ Mid(n).

(f) Mid$(n) < \int_{0}^{11} f(x)\, dx <$ Trap(n).

Solution: a), d), and e)

11. Approximations using Left(10), Right(10), Trap(10), and Mid(10) were made for $\int_a^b f(x)\,dx$. If $f'(x)$ and $f''(x)$ are positive on $[a, b]$, match the results to the rules.

$$6.4267, \qquad 7.2267, \qquad 6.3867, \qquad 5.6267$$

Solution: Trap, Right, Mid, and Left

12. Determine if the improper integral converges or diverges. Show your work/reasoning. If the integral converges, evaluate the integral.

(a) $\displaystyle\int_0^\infty \frac{1}{x^2+4}\,dx$

Solution: The integral converges. Use table formula #24:

$$\int_0^\infty \frac{1}{x^2+4}\,dx = \frac{\pi}{4}.$$

(b) $\displaystyle\int_1^\infty \frac{1}{2^x}\,dx$

Solution: The integral converges.

$$\int_1^\infty \frac{1}{2^x}\,dx = \frac{1}{2\ln 2}.$$

(c) $\displaystyle\int_0^1 \frac{e^x}{(e^x-1)^2}\,dx$

Solution: The integral diverges. Let $u = e^x - 1$:

$$\int_0^1 \frac{e^x}{(e^x-1)^2}\,dx = \infty.$$

(d) $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx$

Solution: The integral converges. Let $u = \cos x$:

$$\int_{\pi/6}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx = 2\sqrt{\frac{\sqrt{3}}{2}}.$$

(e) $\displaystyle\int_1^\infty \frac{dx}{(x-2)^3}$

Solution: The integral diverges.

$$\int_1^\infty \frac{dx}{(x-2)^3} = \int_1^2 \frac{dx}{(x-2)^3} + \int_2^\infty \frac{dx}{(x-2)^3}$$

and the first integral diverges.

(f) $\displaystyle\int_5^\infty \frac{du}{u^2-16}$

Solution: The integral converges. Use table formula #26:

$$\int_5^\infty \frac{du}{u^2-16} = -\frac{1}{8}\ln\left(\frac{1}{9}\right) = \frac{1}{8}\ln(9).$$

13. According to a book of mathematical tables, $\int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. Use this formula and substitution to find

$$\int_m^{\infty} e^{-\left(\frac{x-m}{s}\right)^2} dx.$$

Assume $s > 0$.

Solution: Let $u = \frac{x-m}{s}$ and change the endpoints:

$$\int_m^{\infty} e^{-\left(\frac{x-m}{s}\right)^2} dx = \frac{s\sqrt{\pi}}{2}.$$

14. Suppose f is continuous for all real numbers and that $\int_0^{\infty} f(x)\, dx$ converges. Determine which of the following converge. Explain or show your work clearly. Assume $a > 0$.

(a) $\int_0^{\infty} a \cdot f(x)\, dx$

Solution: The integral converges. Rewrite as

$$\int_0^{\infty} a \cdot f(x)\, dx = a \int_0^{\infty} f(x)\, dx.$$

(b) $\int_0^{\infty} f(ax)\, dx$

Solution: The integral converges. Let $u = ax$.

(c) $\int_0^{\infty} (a + f(x))\, dx$

Solution: The integral diverges. Rewrite as

$$\int_0^{\infty} (a + f(x))\, dx = \int_0^{\infty} a\, dx + \int_0^{\infty} f(x)\, dx.$$

(d) $\int_0^{\infty} f(a + x)\, dx$

Solution: The integral converges. Let $u = a + x$.

15. Determine if the improper integral converges or diverges. Justify your answer.

(a) $\displaystyle\int_2^\infty \frac{d\theta}{\sqrt{\theta^3+2}}$

Solution: The integral converges. Compare with $\displaystyle\int_2^\infty \frac{d\theta}{\theta^{3/2}}$.

(b) $\displaystyle\int_1^\infty \frac{1+\sin^2 x}{(x+3)^3}\,dx$

Solution: The integral converges. Compare with $\displaystyle\int_1^\infty \frac{1}{(x+3)^3}\,dx = \int_4^\infty \frac{1}{u^3}\,du$.

(c) $\displaystyle\int_1^\infty \frac{(1+\sin^2 x)x^2}{x^3+3}\,dx$

Solution: The integral diverges. Compare with $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.

(d) $\displaystyle\int_2^\infty \frac{x^5}{e^{-x}+1}\,dx$

Solution: The integral diverges. Note that $\displaystyle\lim_{x\to\infty} \frac{x^5}{e^{-x}+1} = \infty$. In order for the improper integral to converge, the integrand must approach 0.

16. If the function $f(x)$ satisfies $f(x) > 0$ for $0 < x < 4$, but $\displaystyle\lim_{b\to 4^-} \int_0^b f(x)\,dx = +\infty$, which inequalities would imply that $\displaystyle\int_0^4 g(x)\,dx$ also diverges? (select all that apply).

(a) $g(x) < f(x)$ for $0 < x < 4$

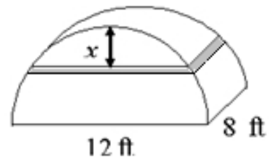
(b) $g(x) < -f(x)$ for $0 < x < 4$

(c) $g(x) > f(x)$ for $0 < x < 4$

(d) $g(x) < f(x)$ for $0 < x < 2$

Solution: (b) and (c)

17. Use the concept of slicing and the variable shown to answer the following about the solid.



(a) Write a formula for the volume of the slice.

Solution: Using the Pythagorean Theorem:

$$\text{volume of slice} \approx 16\sqrt{36-(6-x)^2}\,\Delta x.$$

(b) Write a Riemann sum that approximates the volume of the solid.

Solution: Using the notation of the text:

$$\text{volume of solid} \approx \sum 16\sqrt{36-(6-x)^2}\,\Delta x.$$

(c) Write an integral for the volume of the solid.

Solution:

$$\text{volume} = \int_0^6 16\sqrt{36-(6-x)^2}\,dx.$$

18. Consider the region bounded by $y = -3x + 6$, $y = 3\sqrt{x}$, and the x-axis. Sketch and shade in this region. Set up the integral(s) needed to find the area if we use the following:

(a) slices that are perpendicular to the x-axis.

Solution:

$$\int_0^1 3\sqrt{x}\,dx + \int_1^2 (6 - 3x)\,dx.$$

(b) slices that are perpendicular to the y-axis.

Solution:

$$\int_0^3 \left(\frac{6-y}{3} - \frac{y^2}{9}\right) dy.$$

19. Consider the region bounded by $y = 5e^{-x}$, $y = 5$, and $x = 3$. Find the volume of the solid obtained by rotating the region around the following:

(a) the x-axis

Solution:

$$75\pi - \int_0^3 \pi\left(5e^{-x}\right)^2 dx = \frac{(125 + 25e^{-6})\pi}{2}.$$

(b) the line $y = 5$

Solution:

$$\int_0^3 \pi\left(5 - 5e^{-x}\right)^2 dx = \frac{(75 + 100e^{-3} - 25e^{-6})\pi}{2}.$$

20. Consider the region bounded by $y = x^3$, $y = 8$, and $x = 0$. Find the volume of the solid obtained by rotating the region around the following:

(a) the y-axis

Solution:

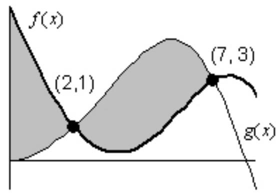
$$\int_0^8 \pi\left(y^{1/3}\right)^2 dy = \frac{96\pi}{2}.$$

(b) the line $x = -2$

Solution:

$$\int_0^8 \pi\left(y^{1/3} + 2\right)^2 dx - 32\pi = \frac{336\pi}{2}.$$

21. Consider the shaded region bounded by $y = f(x)$ and $y = g(x)$ as shown. Set up the integral needed to find the volume of the solid obtained by rotating the region around the x-axis.



Solution:

$$\int_0^2 \left(\pi\,(f(x))^2 - \pi\,(g(x))^2\right) dx + \int_2^7 \left(\pi\,(g(x))^2 - \pi\,(f(x))^2\right) dx.$$

22. Consider the region bounded by the first arch of $y = \sin x$ and the x-axis. Find the volume of the solid whose base is this region and whose cross-sections perpendicular to the x-axis are the following:

(a) squares

Solution:

$$\int_0^{\pi} (\sin x)^2 \, dx = \frac{\pi}{2}.$$

(b) semi-circles

Solution:

$$\int_0^{\pi} \frac{1}{2}\pi \left(\frac{\sin x}{2}\right)^2 dx = \frac{\pi^2}{16}.$$

23. The circumference of a tree at different heights above the ground is given in the table below. Assuming all of the horizontal cross-sections are circular, estimate the volume of the tree.

Height (inches)	0	10	20	30	40	50
Circumference (inches)	26	22	18	12	6	2

Solution: Using the left hand rule:

$$10 \cdot \pi \left(\frac{26}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{22}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{18}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{12}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{6}{2\pi}\right)^2 = \frac{4160}{\pi} \text{ cubic inches.}$$

Using the right hand rule:

$$10 \cdot \pi \left(\frac{22}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{18}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{12}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{6}{2\pi}\right)^2 + 10 \cdot \pi \left(\frac{2}{2\pi}\right)^2 = \frac{2480}{\pi} \text{ cubic inches.}$$

Using the trapezoid rule (average of the left and right hand rules):

$$\frac{3320}{\pi} \text{ cubic inches.}$$

24. Set up, but do not evaluate the integrals needed to find the volumes of the solids.

(a) The solid obtained by rotating the region bounded by $y = x^2$ and $x = y^3$ around the y-axis.

Solution:

$$\int_0^1 \left(\pi \left(y^3\right)^2 - \pi \left(\sqrt{y}\right)^2\right) dy.$$

(b) The solid obtained by rotating the region under $f(x) = \dfrac{1}{x^2+1}$ for $x \geq 0$ around the x-axis.

Solution:

$$\int_0^{\infty} \pi \left(\frac{1}{x^3+1}\right)^2 dx.$$

25. A metallic rod 5 cm in length is made from a mixture of several materials so that its density changes along its length. Suppose the density of the rod at a point x cm from one end is given by

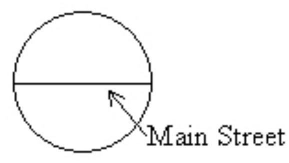
$$\delta(x) = 2 + 0.5\cos x \quad \text{grams per cm of length.}$$

Find the total mass of the rod.

Solution:

$$\int_0^5 (2 + 0.5\cos x)\, dx = 10 + 0.5\sin 5 \text{ grams.}$$

26. Suppose a city is roughly circular with a radius of 8 miles and the density of people can be modeled by some function $\delta(x)$ in people per square mile. Set up an integral to find the total population if x is the distance in miles from:



(a) the center of the city

Solution: $\int_0^8 \delta(x) \cdot 2\pi x\, dx.$

(b) Main Street

Solution: $\int_{-8}^8 \delta(x) \cdot 2\sqrt{64 - x^2}\, dx.$

27. A cylindrical form is filled with slow-curing concrete to form a column. The radius of the form is 10 feet and the height is 25 feet. While the concrete hardens, gravity causes the density to vary so that the density at the bottom is 90 pounds per cubic foot and the density at the top is 50 pounds per cubic foot. Assume that the density varies linearly from top to bottom. Find the total weight (in pounds) of the concrete column.

Solution:

$$\int_0^{25} \left(-\frac{8}{5}h + 90\right) \pi(10)^2 dh = 175,000\pi \text{ pounds.}$$

28. (a) Find a formula for the general term of the sequence

$$\frac{-2}{9}, \frac{4}{16}, \frac{-6}{25}, \frac{8}{36}, \frac{-10}{49}, \ldots$$

Solution: $a_n = \dfrac{(-1)^n \cdot 2n}{(n+2)^2}$

(b) Determine if the sequences converge or diverge. If the sequence converges, find its limit.

$$a_n = \frac{3n^2+2}{2-5n^2} \qquad\qquad b_n = \frac{5(n+1)!}{n^2(n-1)!}$$

Solution:

$$\lim_{n\to\infty} a_n = -\frac{3}{5} \qquad\qquad \lim_{n\to\infty} b_n = 5$$

29. Find the following sums:

(a) $\displaystyle\sum_{k=3}^{10} 3\left(\frac{1}{4}\right)^k$

Solution:

$$\frac{\left(\frac{3}{64}\right)\left(1-\left(\frac{1}{4}\right)^8\right)}{1-\frac{1}{4}} = \frac{65535}{1048576}$$

(b) $\displaystyle\sum_{k=1}^{\infty} 3\left(\frac{1}{4}\right)^k$

Solution: $\dfrac{\frac{3}{4}}{1-\frac{1}{4}} = 1$

30. A 200 mg dose of a particular medicine is given every 24 hours. Suppose 5% of the dose remains in the body at the end of 24 hours. Let P_n represent the amount of medicine that is in the body right before the n^{th} dose is taken. Let Q_n represent the amount of medicine that is in the body right after the n^{th} dose is taken. Express P_n and Q_n in closed-form.

Solution:

$$P_n = \frac{(0.05)(200)\left(1-0.05^{n-1}\right)}{1-0.05} \qquad Q_n = \frac{(200)\left(1-0.05^{n}\right)}{1-0.05} \qquad n = 1, 2, 3, \ldots$$

31. Use the integral test to determine if the series converges or diverges. (You will not always be told which series test to use during the final exam.)

(a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution: The series converges. Use the method of section 7.7 to evaluate the improper integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}$$

(b) $\sum_{n=1}^{\infty} \frac{3n^2 + 2n}{\sqrt{n^3 + n^2 + 1}}$

Solution: The series diverges. Use the method of section 7.7 to evaluate the improper integral:

$$\int_1^{\infty} \frac{3x^2 + 2x}{\sqrt{x^3 + x^2 + 1}} dx = \infty$$

32. Use the ratio test to determine if the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{e^{n+1}}{n^2 2^n}$

Solution: The series diverges.

$$\lim_{n \to \infty} \frac{e(n)^2}{2(n+1)^2} = \frac{e}{2} > 1$$

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{(n!)^2}{(2n)!}$

Solution: The series converges.

$$\lim_{n \to \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

33. What does the ratio test tell us about the convergence or divergence of the following series?

$$\sum_{n=0}^{\infty} \frac{6n}{\sqrt{3 + n^5}}$$

Solution: The Ratio Test gives us 1 and does not tell us anything about the convergence or divergence of the series.

34. Which test(s) could be used to prove that the series $\sum_{n=1}^{\infty} n^{-3/2}$ converges? (select all that apply)

(a) The p series test.

(b) The integral test.

(c) The ratio test.

(d) The comparison test, using the series $\sum_{n=1}^{\infty} n^{-1}$

Solution: a) and b)

35. Indicate whether the following statements are True or False.

(a) If $0 \le a_n \le b_n$ and $\sum a_n$ converges, then $\sum b_n$ converges.

Solution: False

(b) If $0 \le a_n \le b_n$ and $\sum a_n$ diverges, then $\sum b_n$ diverges.

Solution: True

(c) If $\sum a_n$ converges, then $\sum |a_n|$ converges.

Solution: False

(d) If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Solution: True

(e) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

Solution: False

36. Determine the radius of convergence and the interval of convergence (you do not need to investigate convergence at the endpoints):

(a) $\sum_{n=1}^{\infty} \frac{(2n+1)(x+4)^n}{3^{n+1}}$

Solution: The radius of convergence is $R = 3$. The interval of convergence is $(-7, -1)$.

(b) $3 - \frac{3x^2}{2!} + \frac{3x^4}{4!} - \frac{3x^6}{6!} + \frac{3x^8}{8!} + \cdots$

Solution: The radius of convergence is $R = \infty$. The interval of convergence is $(-\infty, \infty)$.

(c) $1 + \frac{(x-1)}{2} + \frac{2!(x-1)^2}{4} + \frac{3!(x-1)^3}{8} + \frac{4!(x-1)^4}{16} + \cdots$

Solution: The radius of convergence is $R = 0$. The series only converges for $x = 1$.

37. Suppose that $\sum_{n=0}^{\infty} C_n(x-2)^n$ converges when $x = 4$ and diverges when $x = 6$. Which of the following are True, False, or impossible to determine?

(a) The power series diverges when $x = -3$.

Solution: True

(b) The power series converges when $x = 1$.

Solution: True

(c) The power series diverges when $x = 5$.

Solution: Impossible to determine

38. Find the Taylor polynomial of degree two about $a = 1$ for the function

$$f(x) = (x + 7)^{2/3}$$

Use your polynomial to find an approximation for $f(2)$.

Solution: $P_2(x) = 4 + \frac{1}{3}(x - 1) - \frac{1}{144}(x - 1)^2$
$f(2) \approx P_2(2) = \frac{623}{144} \approx 4.3264$

39. Suppose $P_2(x) = c_0 + c_1 x + c_2 x^2$ is the second degree Taylor polynomial for a function $f(x)$ where $f(x)$ is always increasing and concave down. Determine the signs of c_0, c_1, and c_2.

Solution: The sign of c_0 cannot be determined, $c_1 > 0$, $c_2 < 0$.

40. Consider the function given by

$$f(x) = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k!}{(2k)!}(x - 3)^k.$$

(a) Find $f(3)$.

Solution: $f(3) = -1$.

(b) Find $f'(3)$.

Solution: $f'(3) = \frac{1}{2}$.

(c) Find $f''(3)$.

Solution: $f''(3) = -\frac{1}{6}$.

(d) Find the Taylor series for $f(3x)$ about $x = 1$. Include an expression for the general term of the series.

Solution: Substitute $3x$ into the series, then simplify:

$$\sum_{k=0}^{\infty} (-1)^{k+1} \frac{k!}{(2k)!} 3^k (x - 1)^k.$$

41. Find the exact value of $\int_0^{1/13} f(x)\, dx$ if

$$f(x) = \sum_{k=0}^{\infty} (k + 1)(x)^k = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

Solution: $\frac{1}{12}$. Integrate term by term. The result is recognizable as the series for $\frac{x}{1 - x}$.

42. Write out the first four nonzero terms of the Taylor series for $\cos(2\theta)$ about $\theta = \dfrac{\pi}{3}$.

Solution: $\cos(2\theta) = -\dfrac{1}{2} - \sqrt{3}\left(\theta - \dfrac{\pi}{3}\right) + \left(\theta - \dfrac{\pi}{3}\right)^2 + \dfrac{2\sqrt{3}}{3}\left(\theta - \dfrac{\pi}{3}\right)^3 + \cdots$.

43. By recognizing each series as a Taylor series evaluated at a particular value of x, find the following sums, if possible. (You will be given a short table of Taylor series for well-known functions during the final.)

(a) $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots$

Solution: $\sin(1)$.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (0.5)^{k+1}}{k+1}$

Solution: $\ln(1.5)$.

(c) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{\pi}{e}\right)^{k+1}$

Solution: Series diverges because $\dfrac{\pi}{e} > 1$.

44. Use the Taylor series for $f(x) = \sin x$ near $x = 0$ to find the value of $g^{(10)}(0)$ for the continuous function

$$g(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Solution: $-\dfrac{1}{11}$. Use the series for $\sin x$ to find the series for $\dfrac{\sin x}{x}$, then consider the term containing x^{11}.

45. Find the Taylor series about 0 for the following functions (include the general term):

(a) $f(x) = x\ln(1+2x)$

Solution:

$$x\ln(1+2x) = 2x^2 - \frac{4x^3}{2} + \frac{8x^4}{4} - \frac{16x^5}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{n+2}}{n+1}$$

(b) $f(x) = e^{-x^2}$

Solution:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

46. Expand $\dfrac{a}{(a+r)^2}$ about 0 in terms of the variable $\dfrac{r}{a}$ where a is a positive constant and r is very small when compared to a.

Solution:

$$\frac{a}{(a+r)^2} = \frac{1}{a}\left(1+\frac{r}{a}\right)^{-2} = \frac{1}{a}\left(1 - 2\left(\frac{r}{a}\right) + 3\left(\frac{r}{a}\right)^2 - 4\left(\frac{r}{a}\right)^3 + \cdots\right)$$

47. Indicate whether the following statements are True or False.

(a) If $f(x)$ and $g(x)$ have the same Taylor polynomial of degree two near $x = 0$, then $f(x) = g(x)$.

Solution: False

(b) The Taylor series for $f(x)g(x)$ about $x = 0$ is

$$f(0)g(0) + f'(0)g'(0)x + \frac{f''(0)g''(0)}{2!}x^2 + \cdots$$

Solution: False

(c) The Taylor series for f converges everywhere f is defined.

Solution: False

48. Write out the first four nonzero terms of the Taylor series about $x = 0$ for

$$f(x) = \int_0^x \tan^{-1}(t)\, dt$$

Solution: The Taylor series for $\tan^{-1}(t)$ at $t = 0$ would be given.

$$f(x) = \int_0^x \tan^{-1}(t)\, dt = \int_0^x \left(t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots\right) dt = \frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} - \frac{x^8}{56} + \cdots$$

49. Identify the equilibrium solution(s) for

$$\frac{dQ}{dt} = Q(Q-1)(Q^2+4)$$

and classify each as stable or unstable. Justify your answer.

Solution: $Q = 0$ stable, $Q = 1$ unstable.

50. Match the following differential equations with one of its solutions.

(a) $\dfrac{dy}{dx} = \sqrt{|y^2 - 4|}$ i) $y = x^3 + 3x^2$

(b) $\dfrac{dy}{dx} = \dfrac{y-5}{x^2y^2+1}$ ii) $y = e^x + e^{-x}$

(c) $\dfrac{dy}{dx} = y - x^3 + 6x$ iii) $y = 5$

(d) $\dfrac{dy}{dx} = y - \ln|y - x| + 1$ iv) $y = e^x + x$

Solution: (a) ii (b) iii (c) i (d) iv

51. Find the values of A and k so that $y(t) = Ae^{kt}$ is a solution to

$$4\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

and passes through the point $(-1, e)$.

Solution: $y(t) = e^{3/4}e^{-(1/4)t}$ or $y(t) = e$

52. Solve the differential equations subject to the initial conditions:

(a) $\dfrac{dy}{dx} = \sqrt{4 - x^2}, \qquad y(1) = -1$

Solution: $y(x) = \dfrac{1}{2}x\sqrt{4-x^2} + 2\arcsin\left(\dfrac{x}{2}\right) - 1 - \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{3}$

(b) $\dfrac{dx}{d\theta} = \cos^2(\theta), \qquad x(\pi) = 1$

Solution: $x(\theta) = \dfrac{1}{2}\sin\theta\cos\theta + \dfrac{\theta}{2} + 1 - \dfrac{\pi}{2}$

(c) $\dfrac{dy}{dt} = 4t(2y - 1), \qquad y(0) = -1$

Solution: Watch for the sign issues when you remove the absolute values:

$$y(t) = -\frac{3}{2}e^{4t^2} + \frac{1}{2}$$

(d) $\dfrac{dy}{dx} = \sqrt{4 - y^2}, \qquad y(0) = 1$

Solution: $y(x) = 2\sin\left(x + \dfrac{\pi}{6}\right)$

53. A particular drug is known to leave a patients system at a rate directly proportional to the amount of the drug in the bloodstream. Previously, a physician administered 9 mg of the drug and estimated that 5 mg remained in the patient's bloodstream 7 hours later.

(a) Write a differential equation for the amount of drug in the patient's bloodstream at time t.

Solution: $\frac{dQ}{dt} = -\alpha Q$ where $\alpha > 0$.

(b) Solve the differential equation in part (a).

Solution: $Q(t) = Ae^{-\alpha t}$.

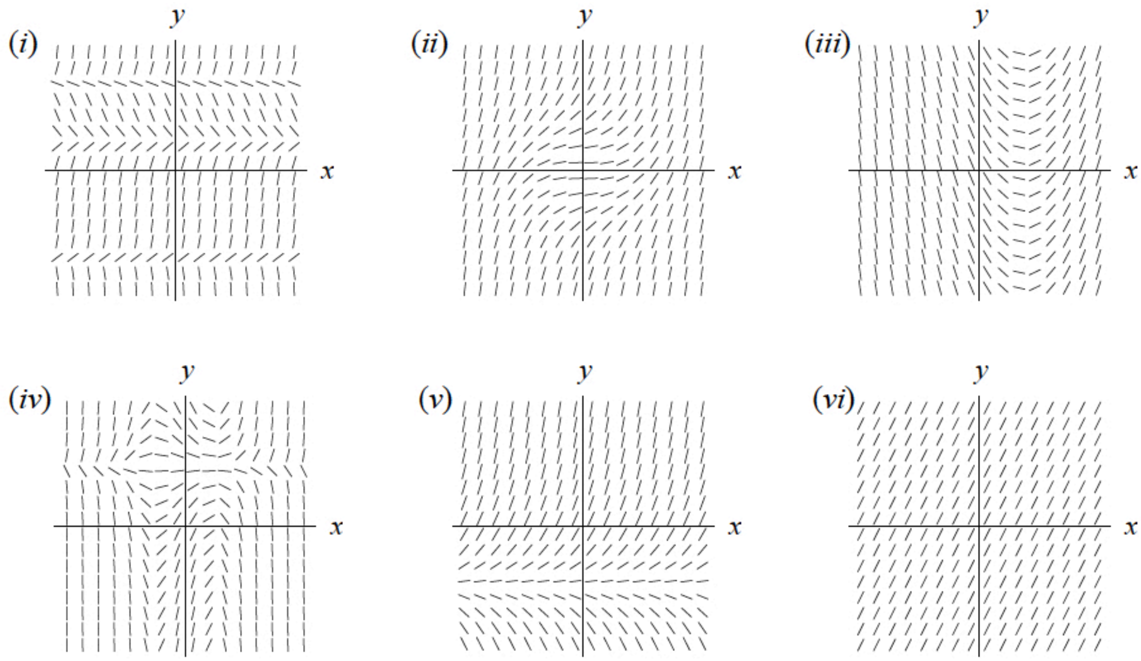
(c) Find the approximate time when the amount of drug in the patient's bloodstream was 0.1 mg.

Solution: $t = \frac{7\ln(90)}{\ln(9/5)} \approx 53.59$ hours.

54. Match the differential equation with the slope field (assume a is a positive constant):

(a) $\dfrac{dy}{dx} = ax^2 + y^2$ (b) $\dfrac{dy}{dx} = x - a$ (c) $\dfrac{dy}{dx} = (y^2 - 4)(y - a)$

(d) $\dfrac{dy}{dx} = (x^2 - 4)(y - a)$ (e) $\dfrac{dy}{dx} = a$ (f) $\dfrac{dy}{dx} = y + a$



Solution: (a) ii (b) iii (c) i (d) iv (e) vi (f) v

55. Dead leaves accumulate on the floor of a forest at a continuous rate of 4 grams per square centimeter per year. At the same time, these leaves decompose continuously at the rate of 60% per year.

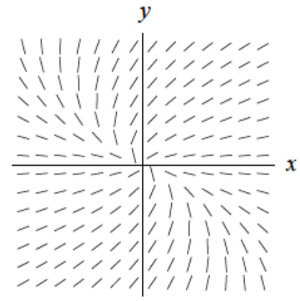
(a) Write a differential equation for the quantity of leaves (in grams per square centimeter) at time t. Solve this differential equation.

Solution: $\dfrac{dL}{dt} = 4 - 0.6L$, $L(t) = \dfrac{Ae^{-0.6t} + 4}{0.6}$.

(b) Find the equilibrium solution and give a practical interpretation. Is the solution stable?

Solution: The stable equilibrium solution is $L = \dfrac{20}{3}$. If we start with 20/3 grams per square centimeter of leaves, we will always have that amount.

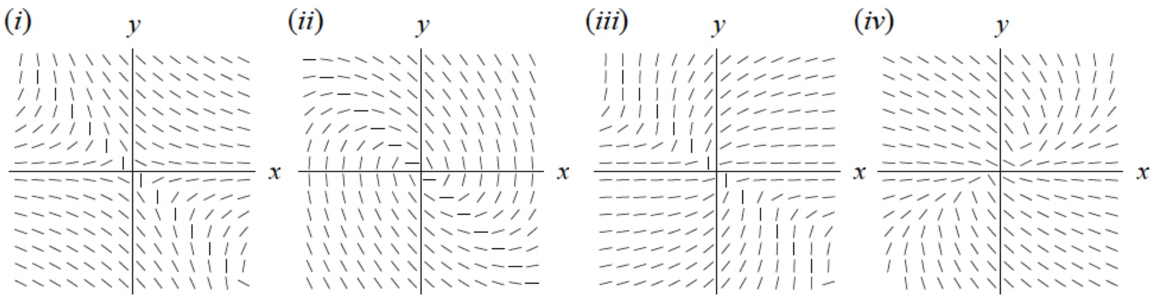
56. The differential equation $\dfrac{dy}{dx} = f(x,y)$ has slope field at the right:



Match the related differential equation with its slope field below:

(a) $\dfrac{dy}{dx} = -f(x,y)$ (b) $\dfrac{dy}{dx} = -\dfrac{1}{f(x,y)}$

(c) $\dfrac{dy}{dx} = (f(x,y))^2$ (d) $\dfrac{dy}{dx} = -f(x,-y)$



Solution: (a) i (b) ii (c) iii (d) iv

57. A room with a southern exposure heats up during the morning. The temperature of the room increases linearly so that it rises 1°F for every 15 minutes. Early in the morning, a cup of coffee with a temperature of 180°F is placed in the room when the room temperature is 60°F. Newtons Law of Cooling states that the rate of change in the temperature of the coffee should be proportional to the difference in temperature between the coffee and the room.

(a) Write a formula for the temperature of the room t minutes after the coffee is placed there.

Solution: $r(t) = \dfrac{1}{15}t + 60.$

(b) Write an initial value problem for the temperature of the coffee as a function of time.

Solution: $\dfrac{dH}{dt} = k\left(H - \left(\dfrac{1}{15}t + 60\right)\right), \quad H(0) = 180$ where $k < 0$.

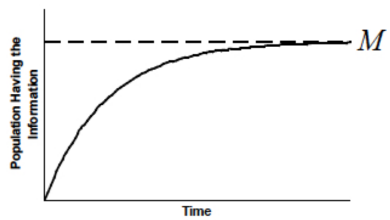
58. The area that a bacteria colony occupies is known to grow at a rate that is proportional to the square root of the area. Assume the proportionality constant is $k = 0.06$. Write a differential equation that represents this relationship. Solve the differential equation.

Solution: $\dfrac{dA}{dt} = 0.06\sqrt{A}, \quad A(t) = (0.03t + c)^2$

59. Two models, based on how information is spread through a population, are given below. Assume the population is of a constant size M.

(a) If the information is spread by mass media (TV, radio, newspapers), the rate at which information is spread is believed to be proportional to the number of people not having the information at that time. Write a differential equation for the number of people *having* the information by time t. Sketch a solution assuming that no one (except the mass media) has the information initially.

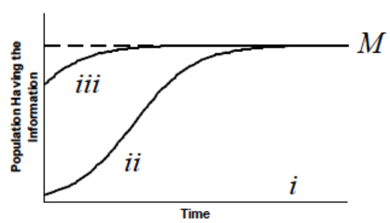
Solution: $\dfrac{dP}{dt} = k\,(M - P), \quad \text{where } k > 0.$



(b) If the information is spread by word of mouth, the rate of spread of information is believed to be proportional to the product of the number of people who know and the number who don't. Write a differential equation for the number of people having the information by time t. Sketch the solution for the cases in which

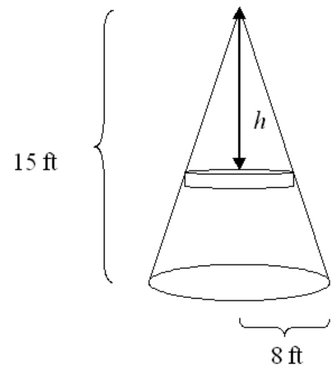
i. no one knows initially,
ii. 5% of the population knows initially, and
iii. 75% of the population knows initially.

Solution: $\dfrac{dP}{dt} = kP\,(M - P), \quad \text{where } k > 0.$



For Classes That Covered Physics (Section 8.5)

1. A tank of water has the shape of a right circular cone shown to the right. In each case, set up the integral for the amount of work needed to pump the water out of the tank under the given conditions. Your integral must correspond to the variable indicated in the picture.
   (The density of water is 62.4 pounds per cubic foot.)



   (a) The tank is full, the tank will be emptied, and the water is pumped to a point at the top of the tank.

   Solution: $\int_0^{15} 62.4(h)\pi \left(\frac{8}{15}h\right)^2 \, dh.$

   (b) The tank is full, the tank will be emptied, and the water is pumped to a point 3 feet above the top of the tank.

   Solution: $\int_0^{15} 62.4(h+3)\pi \left(\frac{8}{15}h\right)^2 \, dh.$

   (c) The tank is full, the water will be pumped until the level of the water in the tank drops to 5 feet, and the water is pumped to a point at the top of the tank.

   Solution: $\int_0^{10} 62.4(h)\pi \left(\frac{8}{15}h\right)^2 \, dh.$

   (d) The initial water level of the tank is 12 feet, the tank will be emptied, and the water is pumped to a point 3 feet above the top of the tank.

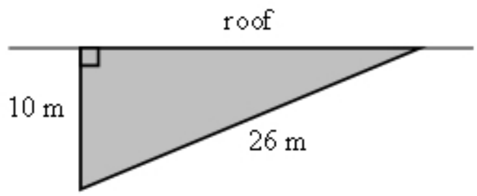
   Solution: $\int_3^{15} 62.4(h+3)\pi \left(\frac{8}{15}h\right)^2 \, dh.$

2. Workers on a platform 45 feet above the ground will lift a block of concrete weighing 500 pounds from the ground to the platform. The block is attached to a chain that weighs 3 pounds per foot. Find the amount of work required.

   Solution: One possibility:

$$500 \cdot 45 + \int_0^{45} 3(45 - x) \, dx = 25,537.5 \text{ foot-pounds.}$$

3. A flag in the shape of a right triangle is hung over the side of a building as shown. It has a total mass of 8 kg and uniform density. Set up the integral needed to find the work done in rolling up the flag to the top of the building. Use $g = 9.8\text{m/s}^2$.



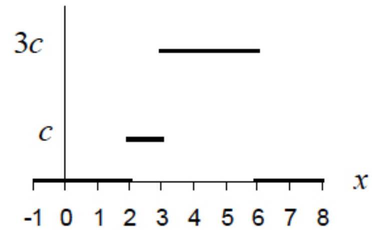
Solution: One possibility:

$$\int_0^{10} 9.8\left(\frac{8}{0.5\cdot 24\cdot 10}\right)(10-h)\left(\frac{12}{5}h\right)\,dh \text{ Joules.}$$

For Classes That Covered Probability (Sections 8.7 and 8.8)

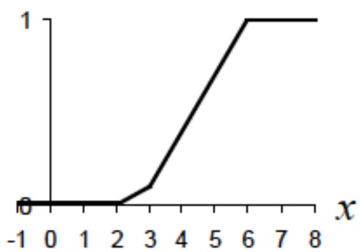
1. (a) Determine if the function at the right is a probability density function or a cumulative distribution function, then find the value of c.

   Solution: Probability density function. $c = \frac{1}{10}$.



   (b) If the function is a probability density function, sketch the cumulative distribution function. If the function is a cumulative distribution function, sketch the probability density function.

   Solution: The cumulative distribution function.



2. Suppose t measures the time (in hours) it takes for students to complete a final exam and no students are allowed to take longer than 3 hours. The density function for t is given by

$$p(t) = \begin{cases} \frac{4}{81}t^3 & 0 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What proportion of the students take more than 2 hours to complete the final exam?

   Solution: $\int_2^3 \frac{4}{81}t^3\,dt = \frac{65}{81}$ or approximately 80.2%.

   (b) Write a piecewise formula for the cumulative distribution function, $P(t)$.

   Solution:

$$P(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{81}t^4 & 0 \le t \le 3 \\ 1 & t > 3 \end{cases}$$

3. The cumulative distribution function for the time (in minutes) between successive calls to a telephone information center is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-0.2t} & t \geq 0 \end{cases}$$

What is the probability that the time between successive calls is between 5 and 10 minutes?

Solution: The probability that the time between successive calls is between 5 and 10 minutes:

$$F(10) - F(5) = e^{-1} - e^{-2} \approx 0.23$$

4. The speeds of cars (in mph) on a road are approximately normally distributed with $\mu = 60$ and $\sigma = 5$. Recall: $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$

(a) According to the model, set up an integral needed to find the probability that a randomly selected car is going more than 60 mph.

Solution: $\displaystyle\int_{60}^{\infty} \frac{1}{5\sqrt{2\pi}} e^{-(x-60)^2/50} dx.$

(b) If the fraction of cars going less than 55 mph is approximately 0.16, approximately what fraction of cars are going more than 65 mph?

Solution: 0.16. Use symmetry of the graph: